A Fixed Point Theorem in T- Metric Space

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Abstract

The notion of T-metric spaces is presented in this article.Two maps on entire T-metric spaces have fixed point theorems that I have presented. By using the same approach to a common fixed point theorem, I have shown that two unovalent mappings in T-metric spaces may be derived.

Keywords: Metric space, fixed point iterations, convergence speed and fixed point.

Introduction

An essential fixed point theorem is the theory of Banach constriction. The scope of this theorem is rather broad.
There are a lot of writers that have looked at fixed point issues for contractive mappings in metric spaces with partial order (see [2]-[4]). We define S-metric spaces and describe some of their characteristics in this article. You may find several publications that employ implicit relations on S-metric spaces, for example, [5]–[9].

For two mappings in full T metric spaces, I shall demonstrate fixed point theorems. For the monovalent situation, I also provide an illustration. First, we will look at the literature and find precise definitions.

Definition 1.1.

Let X be a nonempty set. A function $F : X^3 \to [0,\infty)$ is said to be an F-metric on X, if for each $x,y,z,t \in X$,

M1. $T(x,y,z) \geq 0$,
M2. $T(x,y,z) = 0$ if and only if $x = y = z$, M3.
$T(x,y,z) \leq T(x,x,t) + T(y,y,t) + T(z,z,t)$.

The pair (X,T) is called an T-metric space. [6]

Definition 1.2.

Let (X,T) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball BS(x,r) and closed ball BS[x,r] with center x and radius r as follows, respectively:

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Definition 1.2.

Let (X,T) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows, respectively:

$$B_S[x,r] = \{y \in X : T(y,y,x) \leq r\}. [3]$$

Definition 1.3.

Let (X,T) and $(X^{`},T^{`})$ be two T-metric spaces. A function $f : (X,T) \to (X^{`},T^{`})$ is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X with $T(x_n,x_n,a) \to 0$, $T^{`}(f(x_n),f(x_n),f(a)) \to 0$. I say that f is continuous on X if f is continuous at every point $a \in X$.

Definition 1.4.

Let (X,T) be an F-metric space and $A \subset X$. [11]

1. The set A is said to be an open subset of X, if for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$.

2. The set A is said to be T-bounded if there exists $r > 0$ such that $T(x,x,y) < r$ for all $x,y \in A$.

3. A sequence $\{x_n\}$ in X converges to x if $T(x_n,x_n,x) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists

$n_0 \in N$ such that for $n \geq n_0$, $T(x_n,x_n,x) < \varepsilon$. In this case, we denote by $\lim_{n\to\infty} x_n \to x$ and we say that x is the limit of $\{x_n\} \subset X$.

4. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$T(x_n,x_n,x_m) < \varepsilon \text{ for each } n,m \geq n_0.$$

5. The T-metric space (X,T) is said to be complete if every Cauchy sequence is convergent.

Let r be the set of all $A \subset X$ with $x \in A$ and there exists $r > 0$ such that $B_S(x,r) \subset A$. Then r is a topology on X [8]

Lemma 1.1.

Let (X,T) be an T- Metric Space and suppose that $\{x_n\}$ and $\{y_n\}$ are T-convergent to x,y, respectively. Then I have

$$\limsup_{n\to\infty} T(x_n, z, y_n) \leq T(z,z,x) + T(x,x,y)$$

$$\limsup_{n\to\infty} T(x_n, z, y_n) \leq T(z,z,x)$$

In particular, if $y = x$, t hen I have .

Proof : Let . $\lim_{n\to\infty} y_n \to y$ *and* $\lim_{n\to\infty} x_n \to x$

Then for each $\delta > 0$ there exist $n_1,n_2 \in N$ such that for all $n \geq n_1$

$$T(x_n, x_n, x) < \delta/2$$

and for all $n \geq n_2$

$$. \quad T(y_n, y_n, y) < \delta/4$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by condition of T-metric, I have

$$T(x_n, z, y_n) \leq T(x_n, x_n, x) + T(z, z, x) + T(y_n, y_n, x)$$

$$\leq T(x, x, y) + T(x_n, x_n, x) + T(z, z, x) + 2\,T(y_n, y_n, y)$$

In the above inequality, I get the first desired result for the upper limit $n \to \infty$.. The second conclusion seems clear.

Theorem 1.1 Let (X,T) be an T-metric space. Then the convergent sequence $\{x_n\}$ in X is Cauchy

Theorem 1.2 Let (T,X) be anT -metric space. Then, I have $x,y \in T$ and $T(x,x,y) = T(y,y,x)$

2. Main Results

Theorem 2.1. *Let (X,T) be a complete T-metric space and $F,G : X \to X$ be mappings satisfying the following conditions:*

1. $F(X) \subseteq G(X)$ and either F(X) or G(X) is a closed subset of X,
2. The pair (F,G) is weakly compatible,
3. $T(Fx, Fy, Fz) \leq \psi\,(\max\{T(Gx, Gy, Gz), a_1T(Gz, Fx, Fz), a_2T(Gz,Fy,Fz)\})$ for all $x,y,z \in X$ and $0 < a_1,a_2 < 1$, where $\psi \in \Phi$.

Then the maps F and G have a unique common fixed point. If G is continuous at the fixed point p, then F is also continuous at p.

Note: Φ is reflec the class of all functions $\psi : R^+ \to R^+$ such that ψ is nondecreasing, continuous and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. It is clear that $\psi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and hence,

I have $\psi(t) < t$ for all $t > 0$.

Proof : Let $x_0 \in X$. Define the sequence $y_n = Fx_n = Gx_{n+1}$, $n = 0,1,2,\cdots$ and let $L_{n+1} = T(y_n,y_n,y_{n+1})$.

Then we have L

$$L_{n+1} = T(y_{n-1}, y_{n-1}, y_n)$$
$$= T(Ax_n, Ax_n, Ax_{n+1})$$

$\leq \psi(\max\{T(Gx_n, Gx_n, Gx_{n+1}),\ a_1T(Bx_{n+1}, Ax_n, Ax_{n+1}), a_2T(Gx_{n+1}, Fx_n, Fx_{n+1})\})$

$\leq \psi(\max\{L_n,\ a_1L_{n+1}, a_2 L_{n+1}\})$.

There for $L_{n+1} \leq \psi(L_n)$, $n = 1,2,3,\cdots$.

Depending on this I have,

$$T(y_n, y_n, y_{n+1}) \leq \psi T(y_{n-1}, y_{n-1}, y_n)$$

$$\leq \psi^2 T(y_{n-2}, y_{n-2}, y_{n-1})$$

$$\leq \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\leq \psi^n T(y_0, y_0, y_{+1})$$

Therefore, according to the condition of T- metric (Theorem 2.1.3), for every m > n,

I have $T(y_n, y_n, y_m) \leq 2\, T(y_n, y_n, y_{n+1}) + T(,y_{n+1}, y_{n+1}, y_{n+2})$

$$\leq \sum_{i=n}^{m-3} 2[T(y_i, y_i, y_{i+1}) + T(y_{m-2}, y_{m-2}, y_{m-3})]$$

$$\leq 2[\psi^n(T(y_0, y_0, y_1) + \psi^{n+1} T(y_0, y_0, y_1) + \cdots .. + \psi^{m-2}T(y_0, y_0, y_1)].$$

$$= 2 \sum_{i=n}^{m-3} \psi^i[T(y_0, y_0, y_1)]$$

Therefore ; $\sum_{i=1}^{\infty} \psi^i(s) < \infty$ for all $s > 0$, $T(y_n, y_n, y_m) \to 0$ as $n \to \infty$.

So that each $\delta > 0$, there is $n_0 \in N$ such that for each $m,n \geq n_0$ and $T(y_n, y_n, y_m) < \delta$.

This means that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $q \in X$ such that

$$\lim_{n\to\infty} y_n = q \quad \text{and} \quad q = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_{n+1})$$

Let G(X) be a closed subset of X. Then there exists z ∈ X such that G(z) = q. I prove that F(z) = q.

Since,

$T(Fz, Fz, Fx_n) \leq \psi\, [\max\{T(Gz, Gz, Gx_n),\ a_1 T(Gx_n, Fz, Fx_n),\ a_2 T(Gx_n, Fz, Fx_n)\}]$

$= \psi\, [\max\{T(q, q, y_{n-1}),\ a_1 T(y_{n-1}, Fz, y_n),\ a_2 T(y_{n-1}, Fz, y_n)\}]$

Setting the limit as n → ∞ in the above inequality, I obtain

$T(Fz, Fz, p) \leq \psi\, [\max\{0,\ a_1 \lim_{n\to\infty} sup\, T(y_{n-1}, Fz, y_n),\ a_2 \lim_{n\to\infty} sup\, T(y_{n-1}, Fz,)y_n\}]$

$\leq \psi\, [\max\{0,\ a_1 T(Fz, Fz, q),\ a_2 T(Fz, Fz, q)\}]$

$\leq \max\{a_{1,2}\} T(Fz, Fz, q)$

This shows that $1 \leq \max\{a_{1,2}\}$, it's a contradiction, it's a mistake.

Therefore, from $\psi(t) < t$ for all $t > 0$, I have Fz = Gz = q.

From the poor compatibility of the couple (F; G), I have F(Gz) = G(Fz) and therefore

Fz = Gz .

Let's assume that $Fz \neq z$.

Then

$T(Fq, Fq, Fxn) \leq \psi\, [\max\{T(Gq, Gq, G_{x_n}),\ a_1 T(G_{x_n}, Fq, F_{x_n}),\ a_2 T(G_{x_n}, Fq, F_{x_n})\}]$

$= \psi\, [\max\{T(Gq, Gq, y_{n-1}),\ a_1 T(y_{n-1}, Fq, y_n),\ a_2 T(y_{n-1}, Fq, y_n)\}]$

Taking the upper limit as n → ∞ in the above inequality, I obtain.

$T(F_q, F_q, q) \leq \psi[\max\{a_1 \lim_{n\to\infty} sup\, T(y_{n-1}, F_q, y_n), a_2 \lim_{n\to\infty} sup\, T(y_{n-1}, F_q, y_n), T(F_q, F_q, q)\}]$

$\leq \psi\, [\max\{a_1 T(F_q, F_q, q), a_2 T(F_q, F_q, q), T(F_q, F_q, q)\}]$

$\leq \max\{a_1, a_2\}\, T(F_q, F_q, q)$

Since ψ (t) < t for all t > 0, I have Gp = Fp = p. Thus p is a common fixed point of F and G.

Suppose q' is another common fixed point of F and G. Then, I have

$$T(q, q, q') = T(F_q, F_q, q')$$

$$\leq \psi\, [\max\{ a_1 T(q', q, q'), a_2 T(q', q, q'), T(q, q, q')\}]$$

If $T(q, q, q') \leq \psi \{T(q, q, q')\}$ then $T(q, q, q') \leq \psi \{T(q, q, q')\} < T(q, q, q')$

which one is a contraction.Hence, I have q = q'.

$$\text{If } T(q, q, q') < aT(q', q, q') < aT(q', q, q'),$$

Then

$$T(q, q, q') < aT(q', q, q')$$

$$\leq a\,(T(q, q, q') + 2T(q', q', q')) = a\,T(q, q, q')$$

Where a= max{ a_1, a_2} . This is also a contraction. . Hence, I have q = q'. Thus, q is the unique common fixed point of F and G.

Later, I will prove the continuity of mapping in T-metric spaces.
Let $\{a_n\}$ be any sequence in X such that $\{a_n\}$ is convergent to q.

Then I have

$$T(F_q, F_q, F_{a_n}) \leq \psi\, [\max\{T(Gq, Gq, F_{a_n}), a_1 T(G_{a_n}, Fq, F_{a_n}), a_2 T(G_{a_n}, Fq, F_{a_n})\}]$$

Taking the upper limit as $n \to \infty$ in the above inequality, from the continuity of G at a point q I get

$$\lim_{n\to\infty} \sup T(q, q, F_{a_n}) = \lim_{n\to\infty} \sup T(F_q, F_q, F_{a_n})$$

$$\leq \psi\, [\max\, (a_1 \lim_{n\to\infty} \sup T(G_{a_n}, Fq, F_{a_n}), a_2 \lim_{n\to\infty} \sup T(G_{a_n}, Fq, F_{a_n}), \lim_{n\to\infty} \sup T(F_q, F_q, F_{a_n}))]$$

$$\leq \psi\, [\max\, (a_1 \lim_{n\to\infty} \sup T(q, q, F_{a_n}), a_2 \lim_{n\to\infty} \sup T(q, q, F_{a_n}), 0)]$$

$$\leq \max\{ a_1, a_2\}\, T(q, q, F_{a_n})$$

after this

$$a_1 \lim_{n\to\infty} sup\, T(G_{a_n}, Fq, F_{a_n}) \le a_1 \{ \lim_{n\to\infty} sup\, T(G_{a_n}, G_{a_n}, G_q) + \lim_{n\to\infty} sup\, T(F_q, F_q, G_q) + \lim_{n\to\infty} sup\, T(F_{a_n}, F_{a_n}, G_q)\}$$

And

$$a_2 \lim_{n\to\infty} sup\, T(G_{a_n}, Fq, F_{a_n}) \le a_2 \{ \lim_{n\to\infty} sup\, T(G_{a_n}, G_{a_n}, G_q) + \lim_{n\to\infty} sup\, T(F_q, F_q, G_q) + \lim_{n\to\infty} sup\, T(F_{a_n}, F_{a_n}, G_q)\}$$

I have

$$\lim_{n\to\infty} sup\, T(q, q, F_{a_n}) \le \max\{a_1, a_2\} \lim_{n\to\infty} sup\, T(q, q, F_{a_n})$$

This means that

$$\lim_{n\to\infty} sup\, T(q, q, F_{a_n}) = 0 .$$

Then, I deduce that F is continuous at q.

Corollary : Let (X; T) be a complete T-metric space and A : X →X be a mapping satisfying the following

inequality.

$$T(F_{x_1}, F_{x_2}, F_{x_3}) \le \psi\, [\max\{T(G_{x_1}, G_{x_2}, G_{x_3}),\ a_1 T(G_{x_3}, x_3),\ a_2 T(x_3, F_{x_2}, F_{x_3})\}]$$

for all $x_1, x_2, x_3 \in X$, where $\psi \in \Phi$. Then the mapping F has a unique common fixed point q ∈X .

And, the mapping F is continuous at q.

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